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FREE AND FORCED VIBRATION OF CROSS-PLY LAMINATED COMPOSITE SHALLOW ARCHES

A. A. KHDEIR
Civil Engineering Department, Division of Structural Mechanics, Middle East Technical University, 06531 Ankara, Turkey

and

J. N. REDDY
Mechanical Engineering Department, Texas A & M University, College Station, TX 77843-3123, U.S.A.

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Abstract—A model for the dynamic behavior of a laminated composite shallow arch is developed from shallow shell theory. Linear equations of motion are derived for thin, moderately thick and thick arches. Free vibration of the arch is explored and exact natural frequencies of the third-order, second-order, first-order and classical arch theories are determined for various boundary conditions. A generalized modal approach is presented to solve the dynamic response of cross-ply laminated arches with arbitrary boundary conditions and for arbitrary loadings. The Poisson effect and rotary inertia are incorporated in the formulation of the arch constitutive equation, in the analytical approaches and in the numerical results.

INTRODUCTION

Modern fiber reinforced-composites have been the subject of intense research in the last decade. They can be used in high performance fields, such as military aircraft and space structures. Curved panels are an important structural element of many composite components, which can be modeled as rings or arches. These components frequently are subject to their greatest stresses at dynamic, rather than static loads. This paper addresses the dynamic behavior of cross-ply laminated composite slightly curved beams or shallow arches.

Survey studies on the vibration analysis of arch-type structures had been compiled in (Markus and Nanasi (1981), Laura and Maurizi (1987)). Most of the research deals with isotropic arches. Very few deal with composite sandwich curved beams of three-layers. The finite element method was used by Ahmed (1971), (1972) to study the dynamic response of sandwich curved beams and the effects of shear deformation and rotary inertia on natural frequencies was also investigated. Free and forced vibrations of a three-layered ring were analyzed (DiTaranto (1973), Sagartz (1977)). Hamilton's principle was used to derive the equations of motion. In DiTaranto (1973), analytical expressions were obtained for the response and for the natural frequencies of the ring having an elastic core material, while in Sagartz (1977), a computational technique was developed for transient response evaluation and a companion experimental study was conducted. The dynamic stiffness matrix formulation for curved members of constant section was presented for determining natural frequencies of continuous curved beams undergoing in-plane vibrations (Wang and Guilbert (1981), Issa *et al.* (1987)). In Wang and Issa (1987), the method was extended to forced vibrations of continuous curved beams. An analysis of the vibration of transversely isotropic beams, which have small constant initial curvature was presented in (Rossettos (1971), Rossettos and Squires (1973)). A closed-form general solution to the governing equations was derived. Natural modes and frequencies were determined for both clamped and simply supported end conditions. Closed-form steady-state solutions were presented by Bellow and Semeniuk (1972) for in-plane excitation of thin circular arches subjected to cyclic symmetric and unsymmetric support movement. Arches with pinned-end and clamped-end boundary conditions were considered. The steady-state solutions consist of a series of the free modes of vibration. Natural frequencies of in-plane and out-of-plane vibration based on the Timoshenko beam theory were calculated numerically by (Irie *et al.* (1983), (1982)) for uniform arcs of circular cross-section under all combination of boundary conditions.

Recently, theoretical and experimental works had been conducted to investigate the free and forced vibration of laminated arches. Scrivener (1989) developed a model for the dynamic response of a laminated composite arch from classical shell theory. Several methods of solution were explored, namely the Laplace transformation, the method of particular solution and the eigensolution. The free vibration of the arch was explored and the natural frequencies of the system were determined. The response of the arch to general forcing functions was also considered, by the use of the Fourier transformation technique. Damping through material viscoelasticity and use of the model in evaluation of experimental data were also discussed. Experiments were conducted by (Collins and Johnson (1992)) to measure the three-dimensional static and vibratory response to two-graphite–epoxy, thin walled, open section semi circular frames. The experimental data was used to evaluate a mixed finite element model of the frames that is based on Vlasov-type, thin walled, open section curved beam theory. Most recently a consistent set of equations was derived by (Qatu (1992), (1993)) for the analysis of laminated composite curved beams and closed rings. Equations were developed for thin (Qatu (1992)) and moderately thick curved beams (Qatu (1993)), using the classical and first-order theories, respectively. Natural frequencies for simply supported curved beams were obtained by exact solutions. The Ritz method with algebraic polynomials was used to obtain approximate solutions for arbitrary boundary conditions.

In this paper, an analysis of the vibration of slightly curved cross-ply laminated composite beams is presented. Hamilton's principle is used to derive the equations of motions of four theories. Exact natural frequencies are determined for various end conditions using the state space concept. The combined effects of initial curvature, transverse shear deformation, orthotropicity ratio, stacking sequence and boundary conditions are evaluated and discussed. The dynamic response of the arch to general forcing functions and for arbitrary end conditions is also considered. A generalized modal approach in conjunction with the biorthogonality conditions of the principal modes with respect to the eigenfunctions of the original and adjoint equations, is presented.

EQUATIONS OF MOTION

The equations of laminated shallow shell, Reddy and Liu (1985), theory are reduced to laminated shallow arch by assuming that there is no variation in the y-direction. This requires that all terms containing partial derivative with respect to y equal to zero. Accordingly the displacement field of a laminated shallow arch will be presented as:

$$U(x,z,t) = \left(1+\frac{z}{R}\right)u(x,t) + z\left[\delta_0\frac{\partial w}{\partial x} + \delta_1\phi(x,t)\right] + z^2\,\delta_2\psi(x,t) + z^3\delta_3\left[\phi(x,t) + \frac{\partial w}{\partial x}\right]$$

$$V(x,z,t) = 0$$

$$W(x,z,t) = w(x,t) \tag{1}$$

where U, V and W are the generalized displacements along the x, y and z coordinates, respectively. u, w are the displacements of the arch middle surface in the x and z directions respectively. R is the radius of curvature. ϕ and ψ are displacement component functions. The displacement field in (1) contains as special cases, the displacement fields of the classical arch theory (CAT), ($\delta_0 = -1, \delta_1 = \delta_2 = \delta_3 = 0$), the first-order shear deformation arch theory (FOAT), ($\delta_0 = \delta_2 = \delta_3 = 0, \delta_1 = 1$), the second-order shear deformation arch theory (SOAT), ($\delta_0 = \delta_3 = 0, \delta_1 = \delta_2 = 1$) and the third-order shear deformation arch theory (HOAT), ($\delta_0 = \delta_2 = 0, \delta_1 = 1, \delta_3 = -4/(3h^2)$). Here, h is the total thickness of the arch.

The displacement field of the third-order arch theory provides a parabolic variation of the transverse shear stress with the condition to be zero at the top and bottom surfaces of the arch.

The strain–displacement relations are given by:

$$\varepsilon = \varepsilon^{(0)} + z\varepsilon^{(1)} + z^2\varepsilon^{(2)} + z^3\varepsilon^{(3)}$$
$$\gamma = \gamma^{(0)} + z\gamma^{(1)} + z^2\gamma^{(2)} \tag{2}$$

where

$$\varepsilon^{(0)} = \frac{\partial u}{\partial x} + \frac{w}{R}, \quad \varepsilon^{(1)} = \delta_0 \frac{\partial^2 w}{\partial x^2} + \delta_1 \frac{\partial \phi}{\partial x},$$
$$\varepsilon^{(2)} = \delta_2 \frac{\partial \psi}{\partial x}, \quad \varepsilon^{(3)} = \delta_3 \left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right),$$
$$\gamma^{(0)} = (1+\delta_0)\frac{\partial w}{\partial x} + \delta_1 \phi, \quad \gamma^{(1)} = 2\delta_2 \psi, \quad \gamma^{(2)} = 3\delta_3 \left(\phi + \frac{\partial w}{\partial x} \right). \tag{3}$$

The stress–strain relations for the k-th lamina in the laminate coordinate can be written as:

$$\sigma^{(k)} = Q_{11}^{(k)} \varepsilon$$
$$\tau^{(k)} = Q_{55}^{(k)} \gamma \tag{4}$$

where $Q_{11}^{(k)}$ and $Q_{55}^{(k)}$ are the elastic stiffnesses transformed to the x direction.

Hamilton principle will be used to derive the equations of motion with the associated boundary conditions for the displacement field (1) and constitutive equations (4), we have

$$0 = \int_0^t \left[\int_v (\sigma^{(k)}\,\delta\varepsilon + \tau^{(k)}\,\delta\gamma)\,\mathrm{d}A\,\mathrm{d}x - \int_v \rho^{(k)} (\dot{U}\,\delta\dot{U} + \dot{W}\,\delta\dot{W})\,\mathrm{d}v \right.$$
$$\left. - \int_0^l f(x,t)\,\delta w\,\mathrm{d}x \right] \mathrm{d}t \tag{5}$$

where $f(x, t)$ is the distributed transverse load per unit length.

Introducing the following definition of stress resultants

$$(N, M, L, P) = \int_A \sigma^{(k)}(1, z, z^2, z^3)\,\mathrm{d}A$$
$$(Q, T, S) = \int_A \tau^{(k)}(1, z, z^2)\,\mathrm{d}A. \tag{6}$$

The Euler–Lagrange equations of motion of the shallow arch associated with the displacement field in eqn (1) are:

$$\frac{\partial N}{\partial x} = \bar{I}_1 \ddot{u} + \bar{I}_2 \ddot{\phi} + \bar{I}_3 \ddot{\psi} + \bar{I}_4 \frac{\partial \ddot{w}}{\partial x}$$
$$\delta_1 \frac{\partial M}{\partial x} + \delta_3 \frac{\partial P}{\partial x} - \delta_1 Q - 3\delta_3 S = \bar{I}_2 \ddot{u} + \bar{I}_5 \ddot{\phi} + \bar{I}_6 \ddot{\psi} + \bar{I}_7 \frac{\partial \ddot{w}}{\partial x}$$

$$\delta_2 \frac{\partial L}{\partial x} - 2\,\delta_2 T = \bar{I}_3 \ddot{u} + \bar{I}_6 \ddot{\phi} + \delta_2^2 I_5 \ddot{\psi} + \bar{I}_8 \frac{\partial \ddot{w}}{\partial x}$$

$$(1+\delta_0)\frac{\partial Q}{\partial x} + 3\,\delta_3 \frac{\partial S}{\partial x} - \delta_0 \frac{\partial^2 M}{\partial x^2} - \delta_3 \frac{\partial^2 P}{\partial x^2} - \frac{N}{R} + f$$

$$= I_1 \ddot{w} - \bar{I}_4 \frac{\partial \ddot{u}}{\partial x} - \bar{I}_7 \frac{\partial \ddot{\phi}}{\partial x} - \bar{I}_8 \frac{\partial \ddot{\psi}}{\partial x} - \bar{I}_9 \frac{\partial^2 \ddot{w}}{\partial x^2} \tag{7}$$

with the following associated boundary conditions

Natural B.C.	*Essential B.C.*
N	u
$\delta_1 M + \delta_3 P$	ϕ
$\delta_2 L$	ψ
$\delta_0 M + \delta_3 P$	$\partial w/\partial x$
$(1+\delta_0)Q + 3\,\delta_3 S - \delta_0 \frac{\partial M}{\partial x} - \delta_3 \frac{\partial P}{\partial x} + \bar{I}_4 \ddot{u} + \bar{I}_7 \ddot{\phi} + \bar{I}_8 \ddot{\psi} + \bar{I}_9 \frac{\partial \ddot{w}}{\partial x}$	w

(8)

The inertias are defined by the equations

$$\bar{I}_1 = I_1 + 2\frac{I_2}{R},$$

$$\bar{I}_2 = \delta_1\left(I_2 + \frac{I_3}{R}\right) + \delta_3\left(I_4 + \frac{I_5}{R}\right),$$

$$\bar{I}_3 = \delta_2\left(I_3 + \frac{I_4}{R}\right),$$

$$\bar{I}_4 = \delta_0\left(I_2 + \frac{I_3}{R}\right) + \delta_3\left(I_4 + \frac{I_5}{R}\right),$$

$$\bar{I}_5 = \delta_1^2 I_3 + 2\,\delta_1\,\delta_3 I_5 + \delta_3^2 I_7,$$

$$\bar{I}_6 = \delta_1\,\delta_2 I_4 + \delta_2\,\delta_3 I_6,$$

$$\bar{I}_7 = \delta_0\,\delta_1 I_3 + \delta_0\,\delta_3 I_5 + \delta_1\,\delta_3 I_5 + \delta_3^2 I_7,$$

$$\bar{I}_8 = \delta_0\,\delta_2 I_4 + \delta_2\,\delta_3 I_6,$$

$$\bar{I}_9 = \delta_0^2 I_3 + 2\,\delta_0\,\delta_3 I_5 + \delta_3^2 I_7,$$

$$I_i = b \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \rho^{(k)} z^{(i-1)}\, \mathrm{d}z \quad (i = 1, 2, \ldots, 7). \tag{9}$$

The resultants are related to the total strains by

$$\begin{Bmatrix} N \\ M \\ L \\ P \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{11} & D_{11} & E_{11} \\ B_{11} & D_{11} & E_{11} & F_{11} \\ D_{11} & E_{11} & F_{11} & G_{11} \\ E_{11} & F_{11} & G_{11} & H_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon^{(0)} \\ \varepsilon^{(1)} \\ \varepsilon^{(2)} \\ \varepsilon^{(3)} \end{Bmatrix}, \quad \begin{Bmatrix} Q \\ T \\ S \end{Bmatrix} = \begin{bmatrix} A_{55} & B_{55} & D_{55} \\ B_{55} & D_{55} & E_{55} \\ D_{55} & E_{55} & F_{55} \end{bmatrix} \begin{Bmatrix} \gamma^{(0)} \\ \gamma^{(1)} \\ \gamma^{(2)} \end{Bmatrix} \tag{10}$$

where

$$(A_{11}, B_{11}, D_{11}, E_{11}, F_{11}, G_{11}, H_{11}) = b \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Q_{11}^{(k)}(1, z, z^2, z^3, z^4, z^5, z^6)\,\mathrm{d}z$$

$$(A_{55}, B_{55}, D_{55}, E_{55}, F_{55}) = b \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Q_{55}^{(k)}(1, z, z^2, z^3, z^4)\,\mathrm{d}z. \qquad (11)$$

ANALYTICAL FORMULATION

A generalized modal approach (see, e.g., Singh and Abdelnaser (1992), Khdeir (1994), (1995a, b, c), (1996)) will be derived to solve the equations of motion of laminated composite arch for all boundary conditions. According to this approach, we write eqn (7) in the following form of the four theories:
HOAT

$$\begin{aligned} u'' &= c_1 w' + c_2\phi + c_3 w''' + m_1\ddot{w}' + m_2\ddot{u} + m_3\ddot{\phi} \\ \phi'' &= c_4 w' + c_5\phi + c_6 w''' + m_4\ddot{w}' + m_5\ddot{u} + m_6\ddot{\phi} \\ w'''' &= c_7 w + c_8 w'' + c_9 u' + c_{10}\phi' + m_7\ddot{w} + m_8\ddot{w}'' + m_9\ddot{u}' + m_{10}\ddot{\phi}' + c_0 f \end{aligned} \qquad (12)$$

SOAT

$$\begin{aligned} u'' &= c_1\phi + c_2\psi + c_3 w' + m_1\ddot{u} + m_2\ddot{\phi} + m_3\ddot{\psi} \\ \phi'' &= c_4\phi + c_5\psi + c_6 w' + m_4\ddot{u} + m_5\ddot{\phi} + m_6\ddot{\psi} \\ \psi'' &= c_7\phi + c_8\psi + c_9 w' + m_7\ddot{u} + m_8\ddot{\phi} + m_9\ddot{\psi} \\ w'' &= c_{10} w + c_{11} u' + c_{12}\phi' + c_{13}\psi' + m_{10}\ddot{w} + c_0 f \end{aligned} \qquad (13)$$

FOAT

$$\begin{aligned} u'' &= c_1 w' + c_2\phi + m_1\ddot{u} + m_2\ddot{\phi} \\ \phi'' &= c_3 w' + c_4\phi + m_3\ddot{u} + m_4\ddot{\phi} \\ w'' &= c_5 w + c_6 u' + c_7\phi' + m_5\ddot{w} + c_0 f \end{aligned} \qquad (14)$$

CAT

$$\begin{aligned} u'' &= c_1 w' + c_2 w''' + m_1\ddot{u} + m_2\ddot{w}' \\ w'''' &= c_3 w + c_4 w'' + c_5 u' + m_3\ddot{w} + m_4\ddot{w}'' + m_5\ddot{u}' + c_0 f \end{aligned} \qquad (15)$$

where a prime and dots on a quantity denote the derivative with respect to x and t, respectively. The coefficients in eqns (12), (13), (14) and (15) are presented in Appendix A.

Introducing the following state variables
HOAT

$$y_1 = w,\quad y_2 = w',\quad y_3 = w'',\quad y_4 = u,\quad y_5 = \phi,\quad y_6 = w''',\quad y_7 = u',\quad y_8 = \phi' \qquad (16)$$

SOAT

$$y_1 = w,\quad y_2 = u,\quad y_3 = \phi,\quad y_4 = \psi,\quad y_5 = w',\quad y_6 = u',\quad y_7 = \phi',\quad y_8 = \psi' \qquad (17)$$

FOAT

$$y_1 = w, \quad y_2 = u, \quad y_3 = \phi, \quad y_4 = w', \quad y_5 = u', \quad y_6 = \phi' \tag{18}$$

CAT

$$y_1 = w, \quad y_2 = w', \quad y_3 = w'', \quad y_4 = u, \quad y_5 = w''', \quad y_6 = u'. \tag{19}$$

Equations (12), (13), (14) and (15) in conjunction with (16), (17), (18) and (19) can be combined into a system of first-order equations as:

$$\{y'\} = [M]\{\ddot{y}\} + [K]\{y\} + \{F\}. \tag{20}$$

The nonzero elements of the matrices $[M]$ and $[K]$ are presented in Appendix B. The load vector $\{F\}$ is defined as

$$\{F\}^T = \{0, 0, 0, 0, 0, c_0 f, 0, 0\} \quad \text{for HOAT} \tag{21}$$

$$\{F\}^T = \{0, 0, 0, 0, c_0 f, 0, 0, 0\} \quad \text{for SOAT} \tag{22}$$

$$\{F\}^T = \{0, 0, 0, c_0 f, 0, 0\} \quad \text{for FOAT} \tag{23}$$

$$\{F\}^T = \{0, 0, 0, 0, c_0 f, 0\} \quad \text{for CAT.} \tag{24}$$

The state vector $\{y\}$ will be separated into time and spatial coordinates to solve for the free vibration problem (see, e.g., Khdeir and Reddy (1994), (1990))

$$\{y\} = \{Y(x)\}q(t). \tag{25}$$

To obtain the frequencies and the corresponding eigenfunctions, the generalized coordinate $q(t)$ will be represented as:

$$q(t) = e^{i\omega t} \tag{26}$$

and the eigenfunctions $\{Y\}$ must satisfy the following equation

$$\{Y'\} = [D]\{Y\} \tag{27}$$

where

$$[D] = [K] - \omega^2[M]. \tag{28}$$

The formal solution to eqn (27) is given by:

$$\{Y\} = [C]\begin{bmatrix} e^{\lambda_1 x} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n x} \end{bmatrix}\{l\} \tag{29}$$

where

$$\{l\} = [C]^{-1}\{n\} \tag{30}$$

where $n = 8$ for HOAT and SOAT and $n = 6$ for FOAT and CAT, λ_i are the distinct eigenvalues of matrix $[D]$ while $[C]$ denotes the matrix of eigenvectors of $[D]$.

Substitution of (29) into the desired boundary conditions associated with the edges $x = \pm L/2$ results in a set of homogeneous algebraic equations of the form

$$[B][C]^{-1}\{n\} = \{0\}. \tag{31}$$

For nontrivial solution of eqn (31), the determinant must be zero

$$|B|/|C| = 0. \tag{32}$$

Equations (32) and (29) give the frequencies and the corresponding eigenfunctions, respectively. There are infinite frequencies and the eigenfunctions form a complete set, and eqn (25) will be expressed as:

$$\{y(x,t)\} = \sum_{m=1}^{\infty} \{Y_m(x)\}q_m(t). \tag{33}$$

The boundary conditions for hinged (H) and clamped (C) at the edges $x = \pm L/2$ are:
HOAT

$$H: w = N = M = P = 0$$
$$C: u = \phi = w = \frac{\partial w}{\partial x} = 0 \tag{34}$$

SOAT

$$H: w = N = M = L = 0$$
$$C: u = w = \phi = \psi = 0 \tag{35}$$

FOAT

$$H: w = N = M = 0$$
$$C: u = \phi = w = 0 \tag{36}$$

CAT

$$H: w = N = M = 0$$
$$C: u = w = \frac{\partial w}{\partial x} = 0. \tag{37}$$

Equation (20) is not a self adjoint equation and the eigenfunctions do not form an orthogonal set, therefore we must obtain the eigenfunction of the adjoint of eqn (27) in order to decouple eqn (20). Nayfeh (1981) showed that the adjoint of eqn (27) is

$$\{Z'\} = -[D]^T\{Z\} \tag{38}$$

with the boundary conditions defined according to the following equation

$$\{Z\}^T\{Y\}\Big|_{-l/2}^{l/2} = 0. \tag{39}$$

According to eqn (39) the following boundary conditions will be defined at the edges $x = \pm L/2$:

HOAT

$$\begin{aligned} H&: Z_2 = Z_4 = Z_5 = Z_6 = 0 \\ C&: Z_3 = Z_6 = Z_7 = Z_8 = 0 \end{aligned} \tag{40}$$

SOAT

$$\begin{aligned} H&: Z_2 = Z_3 = Z_4 = Z_5 = 0 \\ C&: Z_5 = Z_6 = Z_7 = Z_8 = 0 \end{aligned} \tag{41}$$

FOAT

$$\begin{aligned} H&: Z_2 = Z_3 = Z_4 = 0 \\ C&: Z_4 = Z_5 = Z_6 = 0 \end{aligned} \tag{42}$$

CAT

$$\begin{aligned} H&: Z_2 = Z_4 = Z_5 = 0 \\ C&: Z_3 = Z_5 = Z_6 = 0. \end{aligned} \tag{43}$$

The solution to eqn (38) is

$$\{Z\} = [H]\begin{bmatrix} e^{-\lambda_1 x} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & e^{-\lambda_n x} \end{bmatrix}\{k\} \tag{44}$$

where $[H]$ denotes the matrix of eigenvectors of $-[D]^T$.

Substitution of eqn (44) into the corresponding boundary conditions defined in eqns (40)–(43) at the edge $x = \pm L/2$ results in a set of homogeneous algebraic equations of the form

$$[E]\{k\} = \{0\}. \tag{45}$$

We solve for the eigenvector $\{k\}$ associated with frequency ω.

To solve for the dynamic response, we substitute eqn (33) in eqn (20), premultiplying by the adjoint eigenfunction $\{Z\}^T$, integrating over the domain and using the following biorthogonality conditions of modes with respect to the eigenfunctions $\{Y_m\}$ and $\{Z_n\}$,

$$-\int_{-l/2}^{l/2} \{Z_n\}^T[M]\{Y_m\}\,dx = M_m\,\delta_{mn} \tag{46}$$

$$\int_{-l/2}^{l/2} \{Z_n\}^T(\{Y'_m\}-[K]\{Y_m\})\,dx = \omega_m^2 M_m\,\delta_{mn} \tag{47}$$

we obtain

$$\ddot{q}_m(t)+\omega_m^2 q_m(t) = \frac{1}{M_m}\int_{-l/2}^{l/2} \{Z_m\}^T\{F_m\}\,dx. \tag{48}$$

For zero initial conditions, the state vector $\{y\}$ will be expressed as

$$\{y_m(x,t)\} = \sum_{m=1}^{\infty} \frac{1}{M_m}\{Y_m\}\int_0^t h_m(t-\tau)\int_{-l/2}^{l/2} \{Z_m\}^T\{F_m(\xi,\tau)\}\,d\xi\,d\tau \tag{49}$$

where $h_m(t-\tau)$ is the impulse response function.

NUMERICAL RESULTS AND DISCUSSION

Exact solution for the fundamental frequencies of symmetric and antisymmetric cross-ply laminated arches has been tabulated in Tables 1, 2, 4 and 5 for various end conditions. In Table 3, frequencies for various modes has been obtained. All of the laminae are assumed to be of the same thickness, density and made of the same orthotropic material properties. The following dimensionless orthotropic material properties are used in the free vibration analysis:

$$E_1 = 40E_2, \quad G_{12} = G_{13} = 0.6E_2, \quad G_{23} = 0.5E_2, \quad \nu_{12} = 0.25.$$

A value of 5/6 is used for the shear correction coefficient of FOAT. The frequencies are nondimensionalised as:

$$\bar{\omega} = \left(\frac{\omega l^2}{h}\right)\sqrt{\frac{\rho}{E_2}}$$

where L is the length of the arch and ρ is the density. Frequencies are determined for hinged–hinged, hinged–clamped and clamped–clamped end conditions. The state space concept has been used to obtain these frequencies. This approach is proved to be efficient, powerful and has no limitations and can be applied to thick, moderately thick and thin laminated arches (see Table 1), using different shearing deformation theories. In Table 1, fundamental frequencies for different values of length of thickness ratios have been displayed for all theories to show the effect of shear deformation. This effect is more pronounced in symmetric cross-ply laminates as well as in clamped boundary conditions. For $L/h = 50$, close results of the fundamental frequency have been achieved by the classical and shearing deformation theories. For this ratio and above, the design engineer can use the classical theory for his analytical computation. Increasing the number of layers for the same thickness will increase the fundamental frequency for antisymmetric cross-ply schemes (see Table 2). As displayed in Table 4, for all boundary conditions, increasing orthotropicity ratio will result in an increase in the dimensionless fundamental frequency for symmetric and antisymmetric laminates. For hinged arches, the initial curvature has a correspondingly smaller influence on the results (Table 5) where for clamped arches the influence is more pronounced.

Table 1. (a) Dimensionless fundamental frequencies, $\bar{\omega}$, for thick, moderately thick and thin antisymmetric cross-ply (0/90) arches ($R/L = 5$)

L/h	Theory	H–H	H–C	C–C
5	HOAT	6.156	8.047	10.751
	SOAT	5.893	7.392	9.768
	FOAT	5.979	7.462	9.799
	CAT	7.174	11.127	16.489
10	HOAT	6.961	10.121	15.667
	SOAT	6.863	9.780	15.078
	FOAT	6.898	9.862	15.201
	CAT	7.288	11.341	18.084
50	HOAT	7.294	11.324	40.282
	SOAT	7.290	11.305	40.250
	FOAT	7.292	11.312	40.261
	CAT	7.310	11.385	40.380
100	HOAT	7.303	11.367	45.441
	SOAT	7.302	11.363	45.362
	FOAT	7.303	11.364	45.389
	CAT	7.307	11.383	45.685

Table 1. (b) Dimensionless fundamental frequencies, $\bar{\omega}$, for thick, moderately thick and thin symmetric cross-ply (0/90/0) arches ($R/L = 5$)

L/h	Theory	H–H	H–C	C–C
5	HOAT	9.190	10.181	12.433
	SOAT	9.798	10.350	12.266
	FOAT	9.187	9.592	11.421
	CAT	17.387	26.933	39.558
10	HOAT	13.586	16.505	21.670
	SOAT	14.121	17.187	22.382
	FOAT	13.642	16.240	21.151
	CAT	17.597	27.381	40.839
50	HOAT	17.427	26.535	57.370
	SOAT	17.472	26.703	57.615
	FOAT	17.433	26.548	57.400
	CAT	17.666	27.520	58.811
100	HOAT	17.608	27.266	93.363
	SOAT	17.619	27.310	93.431
	FOAT	17.609	27.271	93.369
	CAT	17.668	27.524	93.739

Table 2. Variation of dimensionless fundamental frequency, $\bar{\omega}$, of cross-ply laminated arches with the number of layers (NL), $L/h = 10$, $R/L = 5$

NL	Theory	H–H	H–C	C–C
2	HOAT	6.961	10.121	15.667
	CAT	7.288	11.341	18.084
3	HOAT	13.586	16.505	21.670
	CAT	17.597	27.381	40.839
4	HOAT	10.212	13.447	18.540
	CAT	11.712	18.227	27.560
10	HOAT	10.880	14.154	19.240
	CAT	12.667	19.713	29.679

The generalized modal approach, utilizing the state form of the equations and their biorthogonality eigenfunctions, is used to evaluate the dynamic response of arches with different boundary conditions, subjected to arbitrary loading. The numerical applications, are carried out for the case of symmetric and antisymmetric cross-ply laminated shallow arches, whose geometrical and material properties are the same as the one used in the free

Table 3. Dimensionless frequencies, $\bar{\omega}$, for various modes of cross-ply laminated arches using HOAT, $L/h = 10$, $R/L = 5$

Lamination	m	H–H	H–C	C–C
0/90	1	6.961	10.121	15.667
	2	24.580	28.457	32.203
	3	47.635	51.274	54.949
0/90/0	1	13.586	16.505	21.670
	2	36.815	38.895	40.898
	3	60.855	63.039	65.562

Table 4. Variation of dimensionless fundamental frequency, $\bar{\omega}$, of cross-ply laminated arches with orthotropicity ratio (E_1/E_2), $L/h = 10$, $R/L = 5$

Lamination	E_1/E_2	Theory	H–H	H–C	C–C
0/90	2	HOAT	3.337	5.087	7.519
		CAT	3.378	5.257	7.926
	15	HOAT	5.112	7.635	11.604
		CAT	5.248	8.166	12.751
	50	HOAT	7.537	10.856	16.868
		CAT	7.952	12.373	19.805
0/90/0	2	HOAT	3.960	5.986	8.714
		CAT	4.031	6.273	9.399
	15	HOAT	9.635	12.948	17.356
		CAT	10.799	16.803	25.072
	50	HOAT	14.479	17.215	22.673
		CAT	19.669	30.605	45.646

Table 5. (a) Effect of shallowness of antisymmetric cross-ply (0/90) laminated arch on the dimensionless fundamental frequency, $\bar{\omega}$, $L/h = 10$

R/L	Theory	H–H	H–C	C–C
5	HOAT	6.961	10.121	15.667
	SOAT	6.863	9.780	15.078
	FOAT	6.898	9.862	15.201
	CAT	7.288	11.341	18.084
10	HOAT	6.956	10.138	14.193
	SOAT	6.858	9.796	13.516
	FOAT	6.894	9.879	13.644
	CAT	7.282	11.355	16.879
Beam	HOAT	6.945	10.130	13.660
	SOAT	6.847	9.788	12.947
	FOAT	6.883	9.871	13.077
	CAT	7.269	11.342	16.450

Table 5. (b) Effect of shallowness of symmetric cross-ply (0/90/0) laminated arch on the dimensionless fundamental frequency, $\bar{\omega}$, $L/h = 10$

R/L	Theory	H–H	H–C	C–C
5	HOAT	13.586	16.505	21.670
	SOAT	14.121	17.187	22.382
	FOAT	13.642	16.240	21.151
	CAT	17.597	27.381	40.839
10	HOAT	13.607	16.575	20.221
	SOAT	14.142	17.263	20.929
	FOAT	13.663	16.311	19.599
	CAT	17.624	27.490	40.160
Beam	HOAT	13.614	16.599	19.712
	SOAT	14.149	17.288	20.419
	FOAT	13.670	16.335	19.051
	CAT	17.632	27.527	39.931

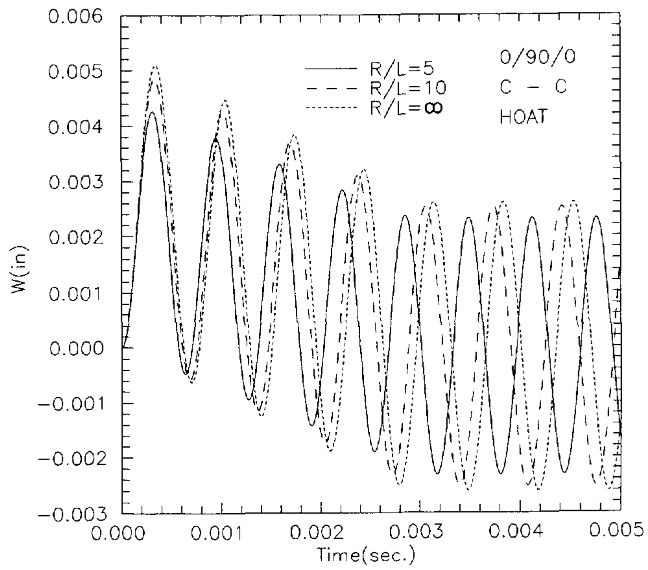


Fig. 1. Effect of shallowness of the arch on the centre deflection of a three-layered (0/90/0) clamped–clamped arch subjected to triangular pulse loading using HOAT.

vibration analysis where $E_2 = 1.0 \times 10^6$ psi, $\rho = 0.00012$ lb-s^2/in^4. A sinusoidal distribution of loading in spatial domain, $f(x, t) = f_0 \cos(\pi x/L)\, p(t)$, is used. The transverse deflection presented in Figs 1–4 is evaluated at the centre ($x = 0$), where the domain of the arch is $-L/2 \leqslant x \leqslant L/2$. Four types of loading in time domain are applied:

1. Triangular pulse loading.

$$p(t) = \begin{Bmatrix} 1 - t/t_1 & 0 \leqslant t \leqslant t_1 \\ 0 & t > t_1 \end{Bmatrix}$$

2. Exponential (blast) loading.

$$p(t) = e^{-\delta t}$$

3. Sine pulse loading.

$$p(t) = \begin{Bmatrix} \sin\left(\dfrac{\pi t}{t_1}\right) & 0 \leqslant t \leqslant t_1 \\ 0 & t > t_1 \end{Bmatrix}$$

4. Step pulse loading.

$$p(t) = \begin{Bmatrix} 1 & 0 \leqslant t \leqslant t_1 \\ 0 & t > t_1 \end{Bmatrix}.$$

The following parameters are used in the numerical computations

$$h = 2 \text{ in}, \quad b = 1 \text{ in}, \quad L = 10h, \quad R = 5L, \quad \delta = 660 \text{ s}^{-1}, \quad t_1 = 0.003 \text{ s}, \quad f_0 = 50 \text{ lb/in}$$

where b is the width of the arch. Zero initial conditions are assumed. The shear correction coefficient for FOAT is set equal to 5/6. The effect of shallowness of the arch on the dynamic response is depicted in Fig. 1 for clamped–clamped end conditions. It is clear that increasing the curvature of the arch will increase the frequency and decrease the dynamic response.

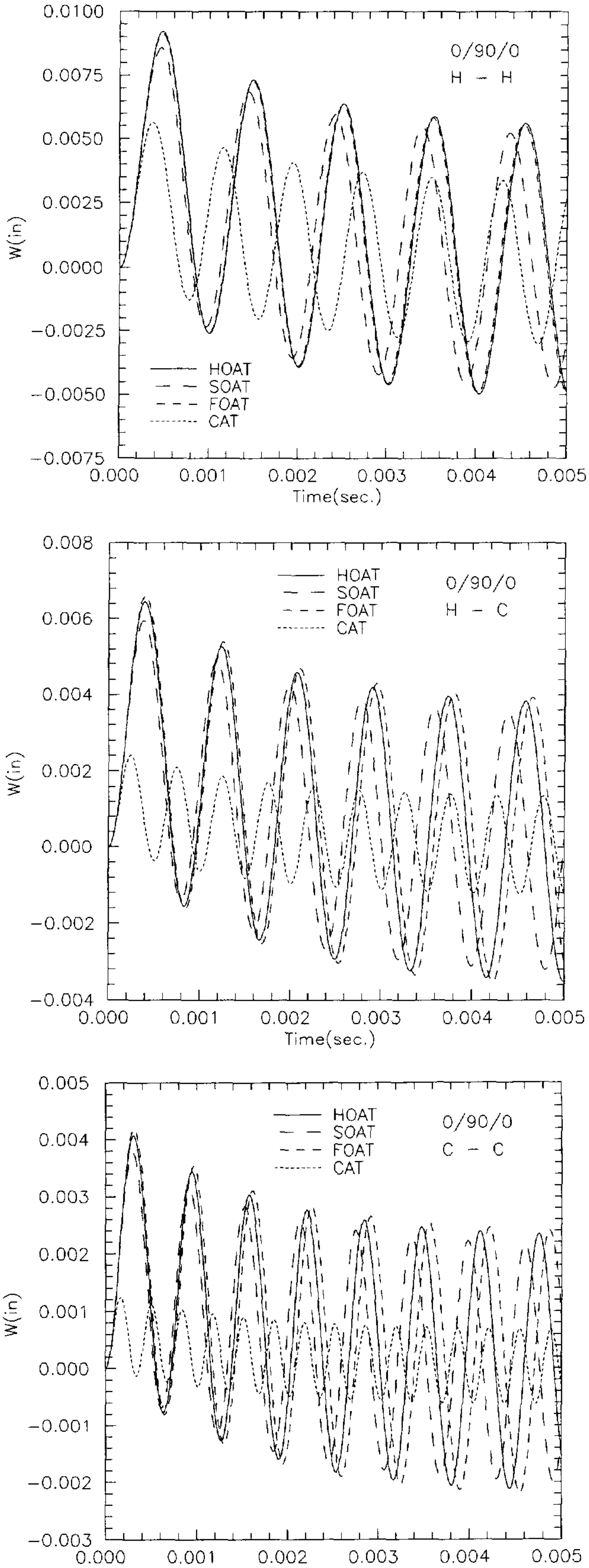


Fig. 2. Centre deflection of three-layered (0/90/0) shallow arch for various boundary conditions (a) hinged–hinged (b) hinged–clamped (c) clamped–clamped using various theories subjected to blast loading.

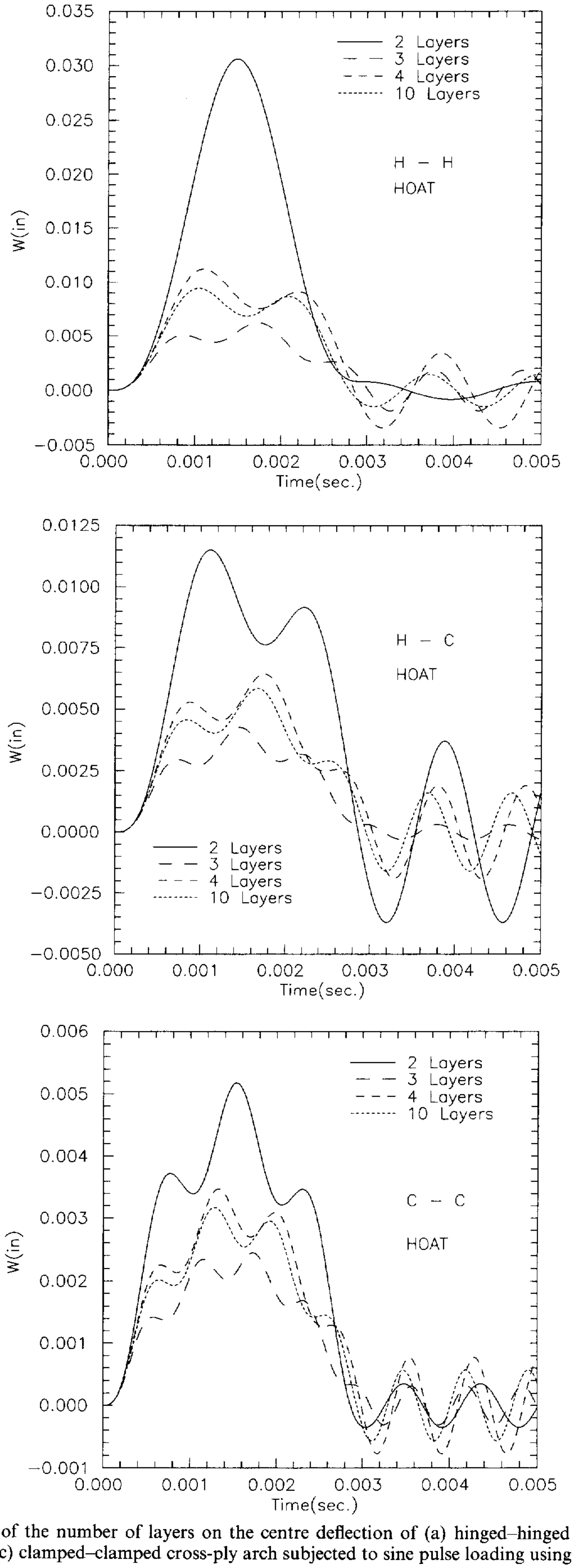
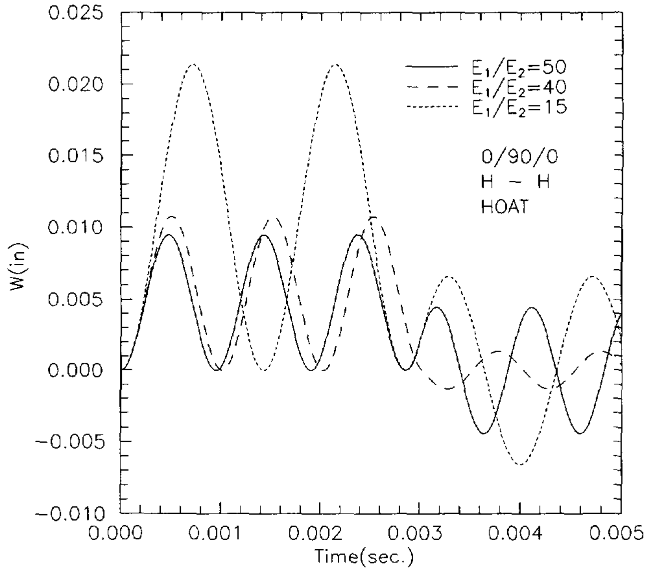
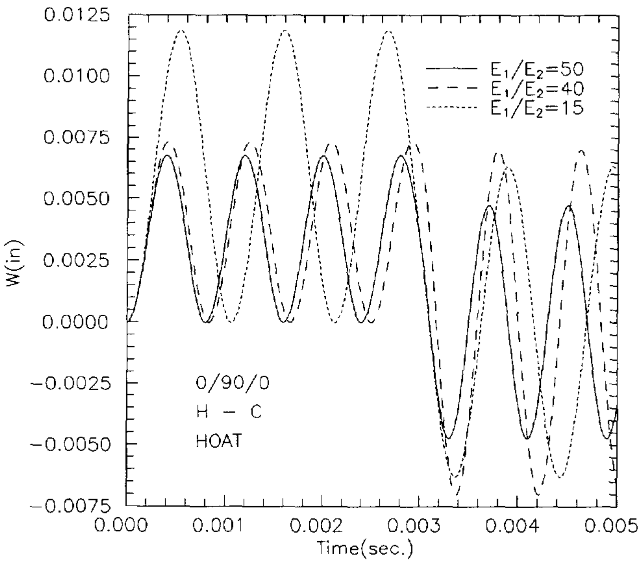


Fig. 3. Effect of the number of layers on the centre deflection of (a) hinged–hinged (b) hinged–clamped (c) clamped–clamped cross-ply arch subjected to sine pulse loading using HOAT.





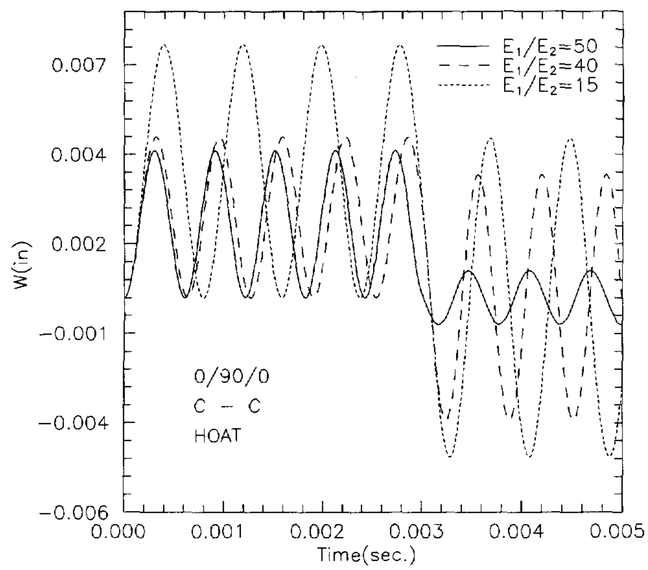


Fig. 4. Effect of orthotropicity ratio on the centre deflection of (a) hinged–hinged (b) hinged–clamped (c) clamped–clamped three-layered (0/90/0) arch subjected to step pulse loading using HOAT.

The comparison between the four theories used in the study is shown in Fig. 2. For all boundary conditions, the centre deflection predicted by HOAT, SOAT and FOAT are clearly higher than those of the CAT. This is due to the fact that the CAT represents the laminate behavior as relatively more stiff. Increasing the number of layers, for the same total thickness, will decrease the amplitude for antisymmetric cross-ply laminated arches, as can be seen from Fig. 3. It is interesting to see that the amplitude for symmetric cross-ply arrangements is smaller than antisymmetric ones. The effect of orthotropicity ratio on the dynamic response is presented in Fig. 4, increasing this ratio will increase the frequency and decrease the response.

It has to be mentioned that the quasi-absolute validity of any strength of materials-type theory can be ascertained by comparison with a mathematical theory of elasticity type approach only.

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APPENDIX A

The coefficients appearing in eqn (12) are:

HOAT

$$c_1 = (e_2e_7 - e_3e_6)e_0,\quad c_2 = e_2e_5e_0,\quad c_3 = (e_2e_8 - e_4e_6)e_0,\quad c_4 = (e_2e_3 - e_1e_7)e_0,$$
$$c_5 = -e_1e_5e_0,\quad c_6 = (e_2e_4 - e_1e_8)e_0,\quad c_7 = e_9c_0,\quad c_8 = (e_{10} - e_4c_1 - e_8c_4)c_0,$$
$$c_9 = -e_3c_0,\quad c_{10} = -(e_4c_2 + e_7 + e_8c_5)c_0,\quad m_1 = (e_6\bar{I}_4 - e_2\bar{I}_7)e_0,$$
$$m_2 = (e_6\bar{I}_1 - e_2\bar{I}_2)e_0,\quad m_3 = (e_6\bar{I}_2 - e_2\bar{I}_5)e_0,\quad m_4 = (e_1\bar{I}_7 - e_2\bar{I}_4)e_0,$$
$$m_5 = (e_1\bar{I}_2 - e_2\bar{I}_1)e_0,\quad m_6 = (e_1\bar{I}_5 - e_2\bar{I}_2)e_0,\quad m_7 = -I_1c_0,$$
$$m_8 = (\bar{I}_9 - e_4m_1 - e_8m_4)c_0,\quad m_9 = (\bar{I}_4 - e_4m_2 - e_8m_5)c_0,$$
$$m_{10} = (\bar{I}_7 - e_4m_3 - e_8m_6)c_0,\quad e_0 = 1/(e_1e_6 - e_2^2),\quad c_0 = 1/(e_4c_3 + e_8c_6 - e_{11})$$

where

$$e_1 = A_{11},\quad e_2 = B_{11} + \delta_3E_{11},\quad e_3 = A_{11}/R,\quad e_4 = \delta_3E_{11},$$
$$e_5 = -3\delta_3(D_{55} + 3\delta_3F_{55}) - (A_{55} + 3\delta_3D_{55}),$$
$$e_6 = D_{11} + 2\delta_3F_{11} + \delta_3^2H_{11},\quad e_7 = e_5 + e_2/R,\quad e_8 = \delta_3F_{11} + \delta_3^2H_{11},$$
$$e_9 = -A_{11}/R^2,\quad e_{10} = -e_5 - 2\delta_3E_{11}/R,\quad e_{11} = -\delta_3^2H_{11}.$$

The coefficients appearing in eqn (13) are:

SOAT

$$\begin{bmatrix} c_1 & c_2 & c_3 & m_1 & m_2 & m_3 \\ c_4 & c_5 & c_6 & m_4 & m_5 & m_6 \\ c_7 & c_8 & c_9 & m_7 & m_8 & m_9 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \\ e_2 & e_3 & e_5 \\ e_3 & e_5 & e_9 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & -e_4 & \bar{I}_1 & \bar{I}_2 & \bar{I}_3 \\ -e_6 & -e_7 & -e_8 & \bar{I}_2 & I_3 & I_4 \\ -e_7 & -e_{10} & -e_{11} & \bar{I}_3 & I_4 & I_5 \end{bmatrix},$$

$$c_{10} = e_{12}/e_6,\quad c_{11} = -e_4/e_6,\quad c_{12} = -e_8/e_6, c_{13} = -e_{11}/e_6,\quad m_{10} = -I_1/e_6,\quad c_0 = 1/e_6$$

where

$$e_1 = A_{11},\quad e_2 = B_{11},\quad e_3 = D_{11},\quad e_4 = A_{11}/R,\quad e_5 = E_{11},\quad e_6 = -A_{55},\quad e_7 = -2B_{55},$$
$$e_8 = B_{11}/R - A_{55},\quad e_9 = F_{11},\quad e_{10} = -4D_{55},\quad e_{11} = D_{11}/R - 2B_{55},\quad e_{12} = -A_{11}/R^2.$$

The coefficients appearing in eqn (14) are:

FOAT

$$c_1 = (e_2e_6 - e_3e_5)e_0,\quad c_2 = e_2e_4e_0,\quad c_3 = (e_2e_3 - e_1e_6)e_0,$$
$$c_4 = -e_1e_4e_0,\quad c_5 = e_7/e_4,\quad c_6 = -e_3/e_4,\quad c_7 = -e_6/e_4,\quad c_0 = 1/e_4,$$
$$m_1 = (e_5\bar{I}_1 - e_2\bar{I}_2)e_0,\quad m_2 = (e_5\bar{I}_2 - e_2I_3)e_0,\quad m_3 = (e_1\bar{I}_2 - e_2\bar{I}_1)e_0,$$
$$m_4 = (e_1I_3 - e_2\bar{I}_2)e_0,\quad m_5 = -I_1/e_4,\quad e_0 = 1/(e_1e_5 - e_2^2)$$

where

$$e_1 = A_{11},\quad e_2 = B_{11},\quad e_3 = A_{11}/R,\quad e_4 = -K^2A_{55},$$
$$e_5 = D_{11},\quad e_6 = B_{11}/R - K^2A_{55},\quad e_7 = -A_{11}/R^2.$$

The coefficients appearing in eqn (15) are:

CAT

$$c_1 = -e_2/e_1,\quad c_2 = -e_3/e_1,\quad c_3 = e_4c_0,\quad c_4 = (e_5 - e_3c_1)c_0,\quad c_5 = -e_2c_0,$$
$$m_1 = \bar{I}_1/e_1,\quad m_2 = \bar{I}_4/e_1,\quad m_3 = -I_1c_0,\quad m_4 = (I_3 - e_3m_2)c_0,$$
$$m_5 = (\bar{I}_4 - e_3m_1)c_0,\quad c_0 = 1/(e_3c_2 - e_6)$$

where

$$e_1 = A_{11},\quad e_2 = A_{11}/R,\quad e_3 = -B_{11},\quad e_4 = -A_{11}/R^2,\quad e_5 = 2B_{11}/R,\quad e_6 = -D_{11}.$$

APPENDIX B

The nonzero elements of the matrix [K] in eqn (20):

HOAT

$$K(1,2) = K(2,3) = K(3,6) = K(4,7) = K(5,8) = 1,$$
$$K(6,1) = c_7, \quad K(6,3) = c_8, \quad K(6,7) = c_9, \quad K(6,8) = c_{10},$$
$$K(7,2) = c_1, \quad K(7,5) = c_2, \quad K(7,6) = c_3,$$
$$K(8,2) = c_4, \quad K(8,5) = c_5, \quad K(8,6) = c_6.$$

SOAT

$$K(1,5) = K(2,6) = K(3,7) = K(4,8) = 1,$$
$$K(5,1) = c_{10}, \quad K(5,6) = c_{11}, \quad K(5,7) = c_{12}, \quad K(5,8) = c_{13},$$
$$K(6,3) = c_1, \quad K(6,4) = c_2, \quad K(6,5) = c_3,$$
$$K(7,3) = c_4, \quad K(7,4) = c_5, \quad K(7,5) = c_6,$$
$$K(8,3) = c_7, \quad K(8,4) = c_8, \quad K(8,5) = c_9.$$

FOAT

$$K(1,4) = K(2,5) = K(3,6) = 1,$$
$$K(4,1) = c_5, \quad K(4,5) = c_6, \quad K(4,6) = c_7,$$
$$K(5,3) = c_2, \quad K(5,4) = c_1, \quad K(6,3) = c_4, \quad K(6,4) = c_3.$$

CAT

$$K(1,2) = K(2,3) = K(3,5) = K(4,6) = 1,$$
$$K(5,1) = c_3, \quad K(5,3) = c_4, \quad K(5,6) = c_5, \quad K(6,2) = c_1, \quad K(6,5) = c_2.$$

The nonzero elements of the matrix [M] in eqn (20):

HOAT

$$M(6,1) = m_7, \quad M(6,3) = m_8, \quad M(6,7) = m_9, \quad M(6,8) = m_{10},$$
$$M(7,2) = m_1, \quad M(7,4) = m_2, \quad M(7,5) = m_3,$$
$$M(8,2) = m_4, \quad M(8,4) = m_5, \quad M(8,5) = m_6.$$

SOAT

$$M(5,1) = m_{10}, \quad M(6,2) = m_1, \quad M(6,3) = m_2, \quad M(6,4) = m_3,$$
$$M(7,2) = m_4, \quad M(7,3) = m_5, \quad M(7,4) = m_6,$$
$$M(8,2) = m_7, \quad M(8,3) = m_8, \quad M(8,4) = m_9.$$

FOAT

$$M(4,1) = m_5, \quad M(5,2) = m_1, \quad M(5,3) = m_2, \quad M(6,2) = m_3, \quad M(6,3) = m_4.$$

CAT

$$M(5,1) = m_3, \quad M(5,3) = m_4, \quad M(5,6) = m_5, \quad M(6,2) = m_2, \quad M(6,4) = m_1.$$